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Full Length Research Paper

The geometric-Poisson exponentially weighted moving control chart with estimated parameters

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In many count data sets, an item may have more than one defect that causes the item to be defective. The Poisson EWMA scheme cannot be used to monitor such defects. The geometric-Poisson exponentially weighted moving average (EWMA) chart has been shown to be an effective scheme to monitor the number of defects over time. In these applications, it is assumed that the process parameters are known or has been accurately estimated. However, in practice, the process parameters are rarely known and must be estimated from reference sample to construct the geometric-Poisson EWMA chart. The performance of the given chart, due to variability in the parameter estimation, might differ from known parameters case. This article explored the effect of estimated parameters on the conditional and marginal performance of the geometric-Poisson EWMA chart. Recommendations about the proper choice of sample-size, smoothing constant and dispersion parameter are made. Results of this study highlight the practical implications of estimation error, and to offer advice to practioners when constructing a phase-I sample.

Keywords: the geometric-Poisson chart; Parameters Estimation; control limits; Average run length; Marginal performance.

INTRODUCTION

Attributes control charts are commonly used to monitor count data in industrial processes. The Poisson is often the standard distribution considered for modeling random counts (e.g. References[1-3]). However, it is not certainly only the underlying distribution for count data (Jackson[4]). Many examples exist of events that occur according to the Poisson distribution, and for each of these Poisson events one or more other events can occur. Such processes include situations where counts tend to occur in clusters or situations where intensity rate of the counts varies randomly over time (see Rice[5]). For example, automobile accidents on a given highway may follow a Poisson rate λ and the number of injuries of each major accident varies according to a certain distribution, say *D*. The compound Poisson distribution is an adequate model for such data sets as discussed by many authors including Gospodinov and Rotondi[6], Hoffman[7], Yu *et al.*[8] and Chen[9].

The geometric-Poisson distribution (a type compound Poisson distribution) is a natural extension of the Poisson distribution where the contribution of each term is distributed according to the geometric distribution. Several real examples of the geometric-Poisson distribution are common in practice. Actual historical data from several US Air Force bases are analyzed using the geometric-Poisson and constant-Poisson distributions (Chen et. al[10]). Sahinoglu[11] employed geometric-Poisson distribution to monitor the number of software failures in software engineering. Robin[12] and Robin et. al[13] used the geometric-Poisson distribution to model overlapping word occurrences. The geometric-Poisson distribution has also been used to model DNA substitution and the total number of fatalities for the accidents by Rosychuk et. al[14] and Ozel and Inal[15]. All these examples demonstrate the usability of the geometric-Poisson distribution in real fields. However, only few works have been done on the use of this distribution in quality control. Chen et. al[10] developed a CUSUM control charts based on the geometric-Poisson distribution to monitor a small sustained shift in wafer manufacturing. Chen[9] proposed an exponentially weighted moving average (EWMA) control chart for monitoring the number of defects over time. The geometric Poisson distribution (compound distribution) was used to develop the proposed chart. The result of study reveals that the proposed EWMA chart, namely geometric-Poisson EWMA chart, is very effective in monitoring and improving quality in production environment than the usual Poisson EWMA chart.

The available literature on the geometric-Poisson EWMA chart has been based on the assumption that the in-control parameters are known or has been accurately estimated. However, in practice, the parameters are generally unknown and the performance of the given chart is affected due to estimation error. Therefore, accurate estimates of the parameters are required to make the statistical performance of the geometric-Poisson EWMA chart reliable. Also, it is important to provide the practioner guidelines such that the effect of estimation error on the geometric-Poisson EWMA chart can be better understood (see, Woodall and Montgomery[16]). A lot of work has been done in the literature on the estimated control limits of the attributes charts. Jensen et. al[17] and Szarka and Woodall[18] provided a detail review on the effect of estimation error on control chart performance. Yang et. al[19], Tang and Cheong[20], Zhang et. al[21] studied the performance of the geometric charts with estimated control limits. They concluded that a large phase I sample size is required for low in-control proportion of nonconforming. Other related studies on the estimated attributes control charts performance include Shu et. al[22], Chakraborti and Human[23], Testik et. al[24], Testik[25], Ozsan et. al[26], Lee et. al[27], Chiu and Tsai[28], Mahmoud and Maravelakis[29], Saleh et. al[30], Zhang et. al[31], Saghir and Lin[3], etc. In fact, the study of the statistical performance of control charts with estimated control limits is a general research issue of importance.

The current article investigates the effect of estimation error on the performance of the geometric-Poisson EWMA chart. The run length properties such as average run length (ARL), the standard deviation of the run length (SDRL), and percentiles of the run length distribution are analyzed using the Markov Chain approach following Saghir and Lin[3] and Chen[9]. The conditional and marginal performances of the run length metrics of the given chart are evaluated. The conditional analysis allows us to understand the effect of overestimating or underestimating the parameters on the run length performance of the chart. While, the marginal performance is useful in providing recommendations regarding minimum sample size, choice of smoothing constant and dispersion parameter. Because it considers the distribution of the estimated parameters and thus accounts for the variability introduced through parameters estimations.

The rest of the article is summarized as follows. In Section 2, the geometric-Poisson EWMA control chart with estimated parameters is given. Section 3 describes different performance evaluation measures. Section 4 evaluates the performance of the estimated control limits of EWMA chart under two different conditions. Finally, the conclusion of the study with discussion is made in Section 5.

The geometric-Poisson EWMA Chart

The geometric-Poisson EWMA chart with known parameters

Let $Y(t)$ be the random variable of the number of defective items, and let $X(t)$ be the random variable of the number of defects that occur up to t, where $t > 0$. According to Chen[9], the density function of the geometric-Poisson compound distribution with parameters λ (rate) and ρ (dispersion) for any $t > 0$ is

$$P[X(t)=0]=e^{-\lambda t},$$

$$P[X(t)=x]=\sum_{y=1}^{x}\frac{(\lambda t)^{y}\,e^{-\lambda t}}{y!}\times\binom{x-1}{y-1}\rho^{x-y}(1-\rho)^{y},\ x=1,2,3,\ldots \qquad (1)$$

where $\lambda>0$, $0<\rho<1$. The expected value and variance of total defects in a fixed unit $t=1$, are derived by Chen et. al[10] and shown as $E[X]=\mu=\frac{\lambda}{1-\rho}$ and $\mathrm{Var}[X]=\sigma^2=\frac{\lambda\,(1+\rho)}{(1-\rho)^2}$, respectively. Clearly, the variance of the geometric Poisson distribution is greater than or equal to the mean. If the variance equals the mean, the geometric Poisson reduces to the Poisson.

Assume that the sequence of $X_1, X_2, \ldots.$ from a repetitive production process of i.i.d. compound geometric-Poisson variables with probability mass function defined in Equation (1). To detect the changes from the in-control mean $\mu=\mu_0$ to an out-of-control mean $\mu=\mu_1$, Chen[9] proposed an EWMA control chart. The EWMA statistic is defined as

Table 1: The mean and percentiles of the sampling distribution of $\hat{\lambda}_0$ at various Phase-I samples, assuming the true values of parameters λ_0 = 2 and ρ =0.20, respectively.

m	The mean and Percentiles of sampling distribution of $\hat{\lambda}_0$.			
	Mean	25th	50th	75th
30	1.9710	2.1588	1.9885	1.8825
50	1.9835	2.1150	1.9905	1.9099
100	1.9900	2.0999	1.9920	1.9401
500	1.9980	2.0518	1.9988	1.9687
1000	1.9990	2.0302	1.9995	1.9790
5000	1.9997	2.0255	1.9998	1.9895

Table 2: Conditional run length performance of the geometric-Poisson EWMA chart with $\hat{\lambda}_0$=2, $\hat{\rho}$ =0.20 and $ARL_0 \approx$ 500.

				25th (overestimation)		50th (Nominal)		75th (underestimation)	
M	*w*	*k*	δ	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*
30	0.05	2.62	0	**165.10**	**133.40**	**498.78**	**491.88**	**2202.3**	**1991.85**
			0.5	19.30	16.29	26.79	21.05	33.12	27.50
			1.0	8.19	5.83	11.15	9.25	20.70	15.30
			1.5	5.11	3.15	7.41	5.05	10.85	8.11
			2.0	3.25	2.85	5.40	3.98	7.65	5.35
	0.10	2.87	0	**182.56**	**169.88**	**499.60**	**496.77**	**1980.20**	**1784.40**
			0.5	22.10	19.90	27.39	22.74	34.88	28.10
			1.0	8.70	6.50	10.17	8.72	19.20	14.11
			1.5	5.52	3.80	6.48	4.12	10.10	7.50
			2.0	3.65	2.99	4.63	3.50	6.50	4.81
	0.20	3.20	0	**196.05**	**188.56**	**503.69**	**497.23**	**1678.90**	**1480.90**
			0.5	27.56	23.55	33.90	29.76	38.55	33.80
			1.0	8.99	6.85	10.67	8.65	20.10	14.00
			1.5	5.60	3.65	6.26	3.97	9.60	6.95
			2.0	3.80	2.72	4.40	2.98	6.05	4.10
100	0.05	2.62	0	**222.56**	**202.10**	**498.78**	**491.88**	**1705.70**	**1480.19**
			0.5	21.22	17.05	26.79	21.05	31.05	25.85
			1.0	9.05	6.05	11.15	9.25	18.95	13.45
			1.5	5.85	3.80	7.41	5.05	9.10	7.65
			2.0	3.90	2.96	5.40	3.98	6.80	4.85
	0.10	2.87	0	**277.56**	**269.55**	**499.60**	**496.77**	**1115.05**	**985.50**
			0.5	23.11	20.25	27.39	22.74	32.70	26.22
			1.0	9.15	6.99	10.17	8.72	18.15	13.05
			1.5	5.91	3.83	6.48	4.12	9.80	6.10
			2.0	3.98	3.10	4.63	3.50	5.70	4.01
	0.20	3.20	0	**305.50**	**297.55**	**503.69**	**497.23**	**990.50**	**799.00**
			0.5	29.50	24.80	33.90	29.76	35.98	32.50
			1.0	9.65	7.05	10.67	8.65	16.50	12.30
			1.5	6.05	3.79	6.26	3.97	8.12	6.01
			2.0	4.01	2.78	4.40	2.98	5.65	3.70
500	0.05	2.62	0	**295.25**	**288.20**	**498.78**	**491.88**	**650.35**	**595.16**
			0.5	23.15	18.20	26.79	21.05	28.85	23.80
			1.0	10.10	7.55	11.15	9.25	15.88	11.20
			1.5	6.56	4.10	7.41	5.05	8.25	6.15
			2.0	4.40	3.36	5.40	3.98	5.85	4.12
	0.10	2.87	0	**353.15**	**340.15**	**499.60**	**496.77**	**580.10**	**510.05**
			0.5	25.55	21.05	27.39	22.74	30.12	24.12
			1.0	9.85	7.25	10.17	8.72	16.20	11.80
			1.5	6.05	3.99	6.48	4.12	7.80	5.85
			2.0	4.09	3.25	4.63	3.50	5.20	3.25
	0.20	3.20	0	**435.56**	**427.85**	**503.69**	**497.23**	**532.10**	**499.05**
			0.5	31.20	26.05	33.90	29.76	34.25	31.05
			1.0	10.05	7.55	10.67	8.65	13.10	10.45
			1.5	6.12	3.85	6.26	3.97	7.65	5.60
			2.0	4.15	2.86	4.40	2.98	5.10	3.50

Table 3: Conditional run length performance of the geometric-Poisson EWMA chart with $\hat{\lambda}_0$=3, $\hat{p}$ =0.20 and $ARL_0 \approx$ 500.

				25th (overestimation)		50th (Nominal)		75th (underestimation)	
M	*w*	*A*	*δ*	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*
			0	**265.50**	**220.90**	**503.03**	**491.96**	**1802.3**	**1690.10**
			0.5	20.50	18.29	28.13	22.82	36.52	32.50
	0.05	2.803	1.0	8.19	7.03	11.56	9.31	25.30	18.30
			1.5	5.11	4.65	7.33	5.85	11.05	9.11
			2.0	3.25	2.90	5.48	3.93	8.55	6.05
			0	**310.56**	**285.88**	**500.64**	**494.05**	**1480.20**	**1288.90**
			0.5	21.50	19.90	28.77	21.76	38.40	34.80
30	0.10	3.447	1.0	9.11	7.97	10.48	9.79	26.50	19.80
			1.5	5.85	4.88	6.34	5.95	12.80	10.50
			2.0	3.70	2.99	4.65	3.90	9.01	6.90
			0	**378.05**	**318.56**	**500.50**	**497.48**	**1270.50**	**1050.20**
			0.5	27.77	22.50	35.06	30.77	39.55	35.70
	0.20	3.878	1.0	9.15	8.05	10.87	9.48	27.20	20.88
			1.5	5.80	4.99	6.50	5.38	13.55	11.12
			2.0	3.90	3.01	4.60	4.01	9.50	7.20
			0	**322.20**	**270.66**	**503.03**	**491.96**	**1050.10**	**990.99**
			0.5	23.90	20.15	28.13	22.82	32.30	28.90
	0.05	2.803	1.0	8.95	7.50	11.56	9.31	20.18	16.10
			1.5	5.88	4.80	7.33	5.85	9.85	7.10
			2.0	4.15	2.70	5.48	3.93	7.05	5.85
			0	**390.44**	**318.62**	**500.64**	**494.05**	**995.15**	**785.50**
			0.5	25.30	18.15	28.77	21.76	33.85	29.50
100	0.10	3.447	1.0	9.45	8.50	10.48	9.79	22.30	17.20
			1.5	5.99	4.50	6.34	5.95	10.10	7.70
			2.0	4.23	2.90	4.65	3.90	7.50	6.00
			0	**410.80**	**385.72**	**500.50**	**497.48**	**880.10**	**689.33**
			0.5	30.33	25.33	35.06	30.77	37.50	33.28
	0.20	3.878	1.0	9.76	8.50	10.87	9.48	23.44	17.81
			1.5	5.95	4.92	6.50	5.38	10.35	8.60
			2.0	4.35	3.25	4.60	4.01	7.60	6.19
			0	**415.15**	**385.15**	**503.03**	**491.96**	**650.35**	**525.16**
			0.5	25.45	20.90	28.13	22.82	29.85	24.15
	0.05	2.803	1.0	9.80	8.20	11.56	9.31	15.20	12.10
			1.5	6.15	4.80	7.33	5.85	8.15	6.50
			2.0	4.86	3.10	5.48	3.93	6.10	4.15
			0	**425.11**	**392.55**	**500.64**	**494.05**	**560.10**	**495.95**
			0.5	26.30	21.10	28.77	21.76	30.12	25.85
500	0.10	3.447	1.0	9.98	8.45	10.48	9.79	16.60	13.00
			1.5	6.05	4.90	6.34	5.95	8.60	6.80
			2.0	4.25	3.15	4.65	3.90	6.45	4.50
			0	**453.85**	**411.50**	**500.50**	**497.48**	**512.10**	**485.85**
			0.5	30.11	27.55	35.06	30.77	36.25	32.10
	0.20	3.878	1.0	10.05	8.50	10.87	9.48	11.95	9.95
			1.5	6.10	4.99	6.50	5.38	7.05	5.85
			2.0	4.45	3.76	4.60	4.01	4.95	4.20

$$Z_i = wX_i + (1-w)Z_{i-1}, \qquad i = 1, 2, \ldots \qquad (2)$$

with $Z_0 = \mu_0$ and $w\ (0 < w \leq 1)$ be the smoothing constant. Since the EWMA can be viewed as a weighted average of all past and current observations, it is very sensitive to the normality assumption. It is therefore an ideal control chart to monitor individual observations and could effectively detect small and moderate changes in the manufacturing processes. For an in-control process, the mean and variance of the EWMA statistic are

$$E(Z_i) = \mu_0 = \frac{\lambda_0}{1-\rho_0}$$

$$Var(Z_i) = \frac{w}{2-w}\left[1-(1-w)^{2i}\right]\left[\frac{\lambda_0(1+\rho_0)}{(1-\rho_0)^2}\right] \qquad (3)$$

The control limits for the EWMA control chart based on the geometric-Poisson compound distribution are defined as:

$$\left.\begin{aligned} LCL &= \frac{\lambda_0}{(1-\rho_0)} - K_l\sqrt{\frac{w}{2-w}\left[1-(1-w)^{2i}\right].\frac{\lambda_0(1+\rho_0)}{(1-\rho_0)^2}} \\ CL &= \frac{\lambda_0}{1-\rho_0} \\ UCL &= \frac{\lambda_0}{(1-\rho_0)} + K_u\sqrt{\frac{w}{2-w}\left[1-(1-w)^{2i}\right].\frac{\lambda_0(1+\rho_0)}{(1-\rho_0)^2}} \end{aligned}\right\}, \qquad (4)$$

Table 4: Conditional run length performance of the geometric-Poisson EWMA chart with $\hat{\lambda}_0$=2, $\hat{\rho}$=0.30 and $ARL_0 \approx 500$.

				25th (overestimation)		50th (Nominal)		75th (underestimation)	
m		*A*		*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*
30	0.05	2.755	0	**172.45**	**156.80**	**496.79**	**489.79**	**2130.30**	**1851.02**
			0.5	21.50	18.19	27.79	21.95	34.52	28.22
			1.0	9.25	6.53	11.63	8.60	21.10	16.30
			1.5	6.35	4.45	7.34	5.03	11.45	8.65
			2.0	4.45	2.85	5.42	3.04	8.25	5.95
	0.10	3.377	0	**191.76**	**179.08**	**498.23**	**492.63**	**1870.50**	**1624.70**
			0.5	23.50	20.50	28.84	22.12	35.78	29.04
			1.0	9.10	7.30	10.75	6.17	20.50	15.50
			1.5	6.22	4.10	6.47	3.12	10.80	8.10
			2.0	4.15	2.79	1.96	1.18	7.50	5.05
	0.20	3.806	0	**208.14**	**191.50**	**499.56**	**494.15**	**1548.60**	**1380.08**
			0.5	28.16	23.95	36.86	32.75	39.45	34.70
			1.0	9.25	7.35	11.74	8.40	21.40	15.40
			1.5	6.10	3.95	6.34	3.74	10.40	7.50
			2.0	4.20	2.96	3.11	1.76	6.75	4.80
100	0.05	2.755	0	**228.55**	**211.99**	**496.79**	**489.79**	**1635.10**	**1490.99**
			0.5	22.32	17.95	27.79	17.95	32.30	26.90
			1.0	9.05	6.55	11.63	5.60	19.75	14.15
			1.5	6.25	4.05	7.34	3.03	10.50	8.35
			2.0	4.50	3.01	5.42	2.04	7.20	5.05
	0.10	3.377	0	**285.50**	**274.05**	**498.23**	**492.63**	**1205.35**	**1005.80**
			0.5	23.95	20.65	28.84	22.12	33.40	27.32
			1.0	9.15	7.15	10.75	6.17	19.11	13.70
			1.5	6.06	2.55	6.47	3.12	10.35	7.05
			2.0	1.05	0.98	1.96	1.18	6.55	5.35
	0.20	3.806	0	**312.10**	**299.88**	**499.56**	**494.15**	**1010.80**	**859.87**
			0.5	29.95	25.09	36.86	32.75	38.18	34.50
			1.0	10.35	7.45	11.74	8.40	17.80	13.05
			1.5	6.85	3.99	6.34	3.74	9.02	7.41
			2.0	4.55	2.98	3.11	1.76	6.15	4.50
500	0.05	2.755	0	**301.25**	**290.60**	**496.79**	**489.79**	**780.45**	**655.87**
			0.5	23.99	18.98	27.79	17.95	30.35	25.40
			1.0	10.76	7.88	11.63	5.60	16.90	10.50
			1.5	7.05	4.74	7.34	3.03	9.95	7.35
			2.0	4.86	3.72	5.42	2.04	7.35	5.12
	0.10	3.377	0	**370.33**	**355.75**	**498.23**	**492.63**	**625.50**	**570.90**
			0.5	26.15	21.56	28.84	22.12	32.50	24.80
			1.0	9.85	6.05	10.75	6.17	16.80	11.80
			1.5	6.05	3.01	6.47	3.12	8.25	5.85
			2.0	1.86	1.06	1.96	1.18	4.20	2.25
	0.20	3.806	0	**445.90**	**437.05**	**499.56**	**494.15**	**560.02**	**509.55**
			0.5	32.20	28.05	36.86	32.75	38.70	33.95
			1.0	10.35	7.85	11.74	8.40	13.75	10.65
			1.5	6.22	3.35	6.34	3.74	8.65	5.60
			2.0	2.96	1.56	3.11	1.76	5.10	2.10

For large values of *i*, the asymptotic limits in equation (4) reduced to

$$\left.\begin{aligned} LCL &= \frac{\lambda_0}{(1-\rho_0)} - K_l\sqrt{\frac{w}{2-w}\cdot\frac{\lambda_0(1+\rho_0)}{(1-\rho_0)^2}} \\ CL &= \frac{\lambda_0}{(1-\rho_0)} \\ UCL &= \frac{\lambda_0}{(1-\rho_0)} + K_u\sqrt{\frac{w}{2-w}\cdot\frac{\lambda_0(1+\rho_0)}{(1-\rho_0)^2}} \end{aligned}\right\} \quad (5)$$

where K_l and K_u are control chart constants and $\frac{\lambda_0}{(1-\rho_0)} = \mu_0$ is the target mean value of the geometric-Poisson compound process. Chen[9] provided the values of $(K_l = K_u = K)$ for various combinations of w, ρ_0, λ_0 and desired in-control ARL using Markov chain approach. The value of the lower control limit *LCL* should be set to zero when it's computed value is less than zero. This is because the quality characteristic of interest X_i is a compound random variable and therefore the EWMA statistic Z_i in equation (2) will be nonnegative. The choice of the smoothing constant w typically depends on how fast a mean shift of given size should be detected. It is generally accepted that smaller values of w are more effective in rapidly detecting smaller mean shifts and vice versa.

Table 5: Marginal performance summaries of the geometric-Poisson EWMA chart for various sizes, shift magnitude and parameters.

λ_0 = 2.0			Smoothing constant *(W)*					
			0.05		0.10		0.20	
ρ	*m*	*δ*	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*
0.20	30	0	**9985.70**	**9865.30**	**1895.20**	**1650.30**	**565.20**	**532.65**
		0.5	1350.60	1125.30	1012.22	890.25	80.52	62.30
		1.0	25.33	14.26	21.15	12.52	13.50	9.60
		1.5	9.35	6.10	8.50	5.80	7.11	4.52
		2.0	6.10	4.25	5.30	3.90	4.62	3.05
	100	0	**1360.50**	**1280.56**	**1105.20**	**998.56**	**552.20**	**520.56**
		0.5	85.52	71.50	70.15	58.33	45.10	34.56
		1.0	19.45	13.52	17.52	11.23	11.90	8.90
		1.5	8.30	5.85	7.10	4.65	6.40	4.02
		2.0	5.65	4.05	5.25	3.70	4.62	3.25
	500	0	**630.52**	**590.25**	**562.56**	**542.60**	**520.56**	**505.60**
		0.5	35.50	28.10	30.25	25.36	36.20	31.02
		1.0	16.12	9.40	12.50	8.90	11.20	8.67
		1.5	7.78	5.55	7.05	4.18	6.32	3.97
		2.0	5.05	3.79	4.79	3.58	4.44	3.01
0.30	30	0	**9735.40**	**9525.90**	**1795.70**	**1480.98**	**540.70**	**512.70**
		0.5	1205.45	1075.90	970.70	805.56	82.12	63.50
		1.0	26.80	15.40	22.25	13.12	14.25	9.94
		1.5	9.95	7.15	9.10	6.05	7.90	5.01
		2.0	6.70	4.60	5.70	4.01	4.93	3.70
	100	0	**1228.90**	**1041.55**	**935.70**	**798.06**	**532.50**	**507.56**
		0.5	82.05	67.52	71.75	60.33	44.70	35.76
		1.0	20.75	13.58	18.42	13.23	12.50	9.10
		1.5	9.05	5.95	7.95	4.96	6.92	4.22
		2.0	6.07	4.35	5.95	3.80	4.71	3.42
	500	0	**615.80**	**565.95**	**543.65**	**542.60**	**516.98**	**503.05**
		0.5	36.10	29.05	31.67	26.22	38.07	32.50
		1.0	16.97	10.25	13.01	9.10	11.82	8.97
		1.5	8.01	5.98	7.80	4.85	6.80	4.01
		2.0	5.85	3.85	4.92	3.70	4.50	3.15

After setting the control limits for the negative binomial EWMA chart, the EWMA statistic given in equation (2) is plotted against each *i*. For an in-control process, all of the Z_i's should lie inside the control limits whereas for an out-of-control process is signaled by one or more of the Z_i's which exceeds the LCL and UCL.

The geometric-Poisson EWMA chart when parameters are unknown

The control chart constant K, for the geometric-Poisson EWMA control statistic can be calculated using the Markov chain approach if λ_0 and ρ_0 are known. However, when λ_0 and ρ_0 are unknown, their values must be calculated prior to any calculations. The method-of-moments estimators are generally used to estimates λ_0 and ρ_0 from a phase I samples. We have the method-of-moment estimates (see Chen et. al[10])

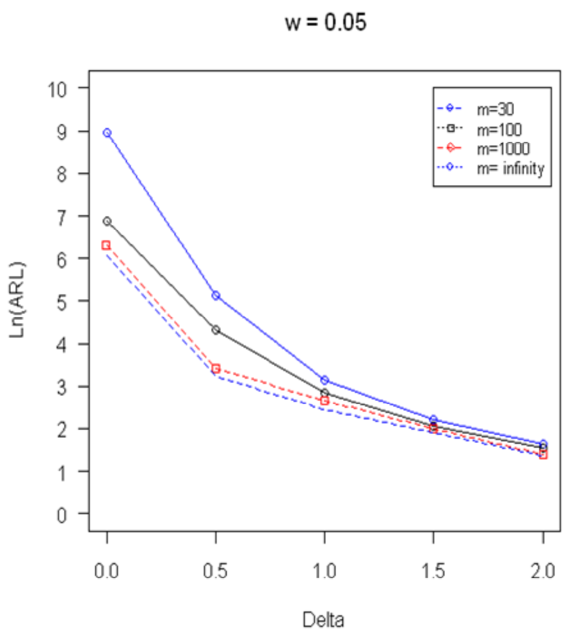
$$\hat{\lambda}_0 = \frac{2(\bar{X})^2}{S^2+\bar{X}} \ , \quad \hat{\rho}_0 = \frac{S^2-\bar{X}}{S^2+\bar{X}} \, . \qquad (6)$$

Here, $\bar{X} = \sum_{i=}^{m} X_i/m$ and $S^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2/m - 1$ are the sample mean and variance, respectively for initial samples of size *m*. The sample variance S^2 must be greater than sample average $\bar{X}$, so that the value of ρ is positive. Therefore, the average number of non-conformities based on moment estimate is $\hat{\mu}_0 = \bar{X}$ and the central limit ensures that its sampling distribution is approximately normal with mean μ_0 and variance μ_0/m. Accordingly, these estimates are used in any of the run length calculations for the geometric-Poisson EWMA chart.

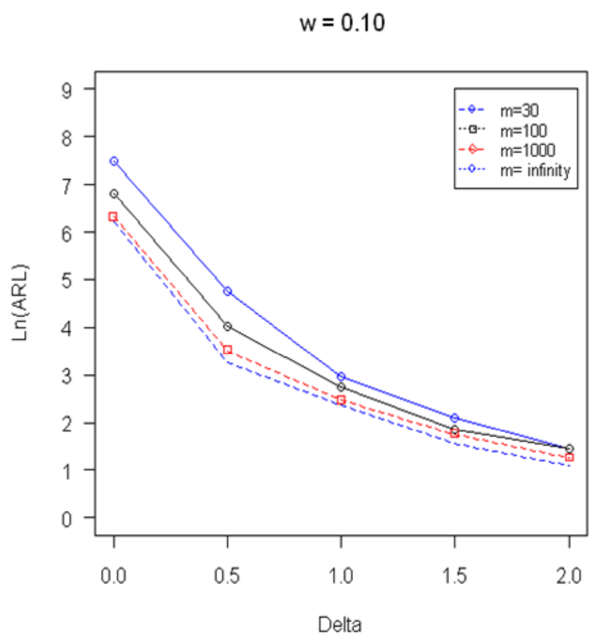
The estimated control limits for the given EWMA chart based on MM estimates are defined as:

$$\left.\begin{aligned} \hat{h}_l(m) &= LCL = \frac{\hat{\lambda}_0}{(1-\hat{\rho}_0)} - K_l\sqrt{\frac{w}{2-w}\cdot\frac{\hat{\lambda}_0(1+\hat{\rho}_0)}{(1-\hat{\rho}_0)^2}} \\ \hat{h}_c\,(m) &= CL = \frac{\hat{\lambda}_0}{(1-\hat{\rho}_0)} \\ \hat{h}_u(m) &= UCL = \frac{\hat{\lambda}_0}{(1-\hat{\rho}_0)} + K_u\sqrt{\frac{w}{2-w}\cdot\frac{\hat{\lambda}_0(1+\hat{\rho}_0)}{(1-\hat{\rho}_0)^2}} \end{aligned}\right\}, \qquad (7)$$

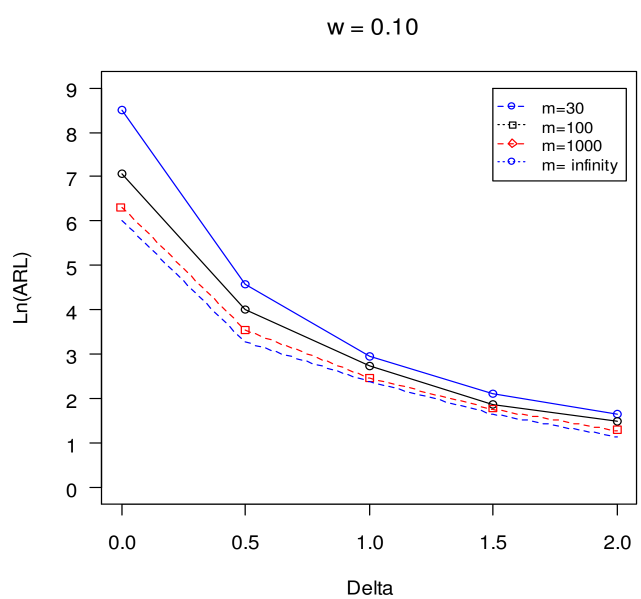
The aim of this article is to determine the effect of the phase I sample on the geometric-Poisson EWMA chart's performance. Statistical properties of a EWMA control chart are usually evaluated in terms of average e run length (ARL), which is the mean of run length (RL) distribution. The ARL of a control charting procedure is defined as the expected number of sampling stages until



(a)

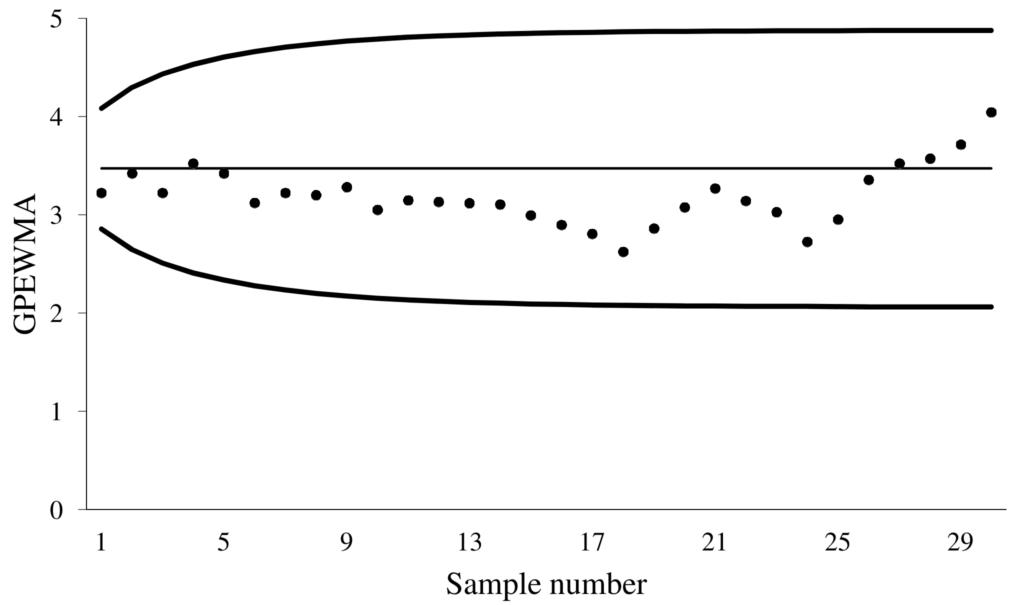


(b)

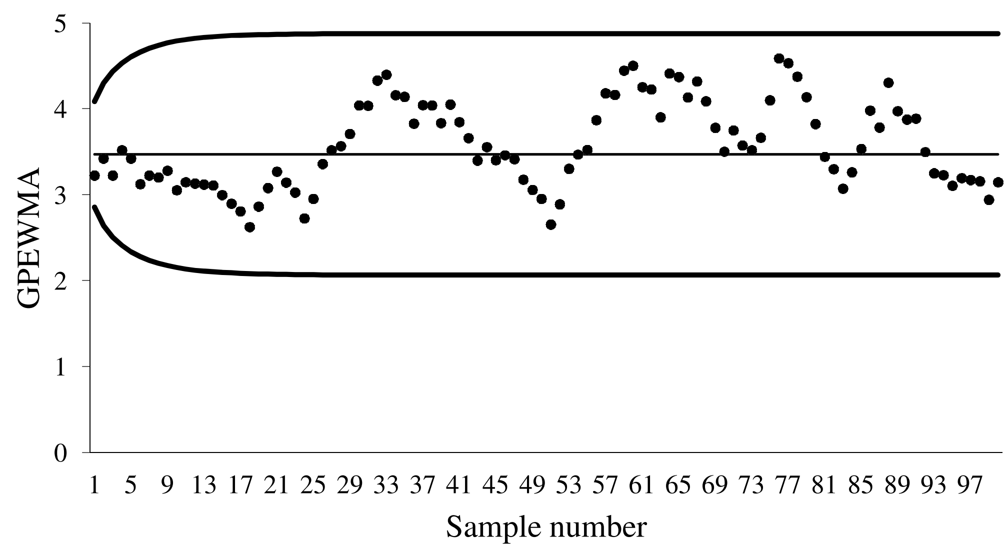


(c)

Fig.1. ARL comparison of the estimated control limits for $\lambda_0 = 3$, $\rho_0 = 0.20$ and (a) $w = 0.05$, (b) $w = 0.10$ and (c) $w = 0.20$ at various samples.



(a)



(b)

Fig. 2. The geometric Poisson EWMA chart for simulated data set using (a) first 30 and (b) all 100 observations at $w = 0.10$.

an out-of-control condition is signaled. An effective and efficient control chart can provide a desired ARL. More specifically, the ARL of a control scheme should be large when a process is in control and small when a shift occurred. In the following section, some information about the calculations will be provided.

Run Length Distribution with Estimated Parameters

Chen (2012) used a Markov chain approach (proposed by Brook and Evans[32]) to study the run length distribution of a geometric-Poisson EWMA chart. In this section, we will extend the Markov chain approximation for assessing the performance of geometric-Poisson EWMA Chart when the estimated parameters differ from their actual values. Similar to Zhang et. al[31], we have considered both conditional and marginal run length properties. We have provided the Markov chain method and the equations used to obtain the run length properties in the coming sub-sections.

The Markov Chain Approach

Suppose D is the number of defects, then the EWMA of D is

$Z_i = wD_i + (1-w)Z_{i-1}$, $i = 1, 2, \ldots$ (8)

The corresponding EWMA control scheme would signal, if $Z_i > \hat{h}_u$ or $Z_i < \hat{h}_u$ and a remedy action should be taken. To visualize the transitioning process, the decision interval $[\hat{h}_l, \hat{h}_u]$ is divided into N subintervals as explained in Chen[9]. The transition of Z_i in the interval $[\hat{h}_l, \hat{h}_u]$ is a random walk and the ith subinterval is the ith state, denoted by E_i, and is represented by the midpoint S_i.

When Z_i falls within the decision interval, then the process is declared to be in-control state. On the other hand, if Z_i moves outside the control limits (above $\hat{h}_u$ or below $\hat{h}_l$), then the process enters to the out-of-control status. Thus, the $(N+1)^{st}$ state is absorbing and represents the out-of-control region.

Let P_{ij} denotes the probability of transition from state i to state j in one step. Then, the transition probability matrix, $\boldsymbol{P}$, is defined as

$$\boldsymbol{P} = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \ldots & P_{0.N+1} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots & P_{1,N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \ldots & P_{i.N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R} & (\boldsymbol{I}-\boldsymbol{R})\boldsymbol{u} \\ \boldsymbol{0} & 1 \end{pmatrix}. \quad (9)$$

Note that all rows sum to unity, and that the last row consists of zeros, except with the last element that is equal to 1 because E_{N+1} is an absorbing state. In addition, the matrix $\boldsymbol{R}$ is a $N \times N$ matrix including the probabilities of moving from one transient state to another, $\boldsymbol{I}$ in an identity matrix of order $N \times N$, $\boldsymbol{u}$ is a $(N \times 1)$ vector of ones, and the $(\boldsymbol{I}-\boldsymbol{R})\boldsymbol{u}$ vector includes the transition probabilities of moving from one transient state to an absorbing state. The transition probabilities for the Markov chain are determined as follows:

$P_{i,1} = \Pr(E_i \to E_1)$, $P_{i,j} = \Pr(E_i \to E_j)$, $P_{i,N+1} = \Pr(E_i \to E_{N+1})$,

$P_{N+1,i} = 0$, $P_{N+1,N+1} = 1$.

where

$$P_{ij} = P[L_j < Z_t < U_j \,|\, Z_{t-1} = S_i = \hat{h}_l + \tfrac{(2i-1)(\hat{h}_u - \hat{h}_l)}{2N}]$$

This transition probability, P_{ij}, can be written as

$$P_{ij} = P\left[\begin{array}{l} \frac{\hat{\lambda}_0}{(1-\hat{\rho}_0)} - K_l\sqrt{\frac{w}{2-w}.\frac{\hat{\lambda}_0(1+\hat{\rho}_0)}{(1-\hat{\rho}_0)^2}} + \frac{(\hat{h}_u-\hat{h}_l)}{2Nw}\{2(j-1)-(1-w)(2i-1)\} \\ < D_i < \frac{\hat{\lambda}_0}{(1-\hat{\rho}_0)} + K_l\sqrt{\frac{w}{2-w}.\frac{\hat{\lambda}_0(1+\hat{\rho}_0)}{(1-\hat{\rho}_0)^2}} + \frac{(\hat{h}_u-\hat{h}_l)}{2Nw}\{2j-(1-w)(2i-1)\} \end{array}\right]$$

$0,1,2,\ldots N.$ (10)

Conditional performance of the run length distribution

The run length of the geometric-Poisson EWMA chart is the number of steps taken starting from the initial state E_1 to reach the absorbing state □$_{□+1}$. Using the Markov chain approach, the approximate ARL and SDRL performance measures are computed as follows:

□□□ = (□ − □)$^{-□}$□ ,
(11)

□□□□ = {ARL + □ [(□ − □)$^{-□}$ - □]ARL − ARL$^{□}$}$^{□/□}$,(12)

where each of □□□ and □□□□ is a □ × 1 vector including ARLs and SDRLs corresponding to all possible states. Assuming that □ is an odd number, then the ((□ + 1)/2)□□ elements of these vectors correspond to the zero-state ARL and SDRL.

The percentile of the run length distribution is another important performance measure of the control charts. The percentiles of the run length distribution may be determined using the cumulative probability for run length. Let □$_□$ be denote the □ × 1 cumulative probability vector, where each of the N entries is for one starting value Z_0 and the index r = 1, 2, … represents a value of the run length. Then

□$_□$ = (□ − □$^{□}$)□ (13)

The cumulative probability times 100 give the percentile corresponding to the run length value r. For example, the 30th percentile for the case of □$_0$ = $\hat{□}_0$ is the smallest value of r for the middle entry of □$_□$ being greater than or equal to 0.3.

Marginal performance of the run length distribution

The marginal performance can be obtained by integrating the conditional performance measures with respect to the density of parametric space as shown in the following equations:

$$□□□_{□□□□□□□□□} = \int_{-\infty}^{\infty} □□□ \; □(□)□□ \qquad (14)$$

$$□□□□_{□□□□□□□□□} = \int_{-\infty}^{\infty} □□□□ \; □(□)□□ \qquad (15)$$

$$□□□□□□□□□□_{□□□□□□□□□} = \int_{-\infty}^{\infty} □□□□□□□□□□ \; □(□)□□ \qquad (16)$$

where the □(□) is a approximated normal distribution of random variable □ . The marginal performance measures are weighted averages of the conditional performance over all the values that the estimation may yield for the in-control mean $□_0$. These integrals can be solved using a numerical integration procedure. In our calculations, we have followed the approach of Ozsan et. al[26] and used the Simpson's quadrature method in Matlab.

RESULTS AND DISCUSSION

In this section, we have evaluated the conditional and marginal run length performance of the geometric-Poisson EWMA control chart when the control limits are estimated. The conditional performance is summarized in Section 4.1 and marginal performance is provided in Section 4.2. All the computations are done in Matlab.

Conditional Performance of the geometric-Poisson EWMA chart

In reality, the geometric-Poisson EWMA chart is often constructed by using the in-control estimated parameters obtained from a Phase I study. The question of how the under study chart would perform in application is of interest for the researchers and practioners. Although it is assumed that the true process mean is unknown, hypothetical cases of estimation may be considered to provide insight of estimation. This will be helpful to know the best/worst case scenarios of estimation. The hypothetical values for the estimation error are obtained through evaluating the percentiles of the sampling distribution of estimated mean. Three different situations are considered: the 25th percentile case (corresponding to *overestimation* of the in-control process mean $\hat{□}_0$), the 50th percentile case (corresponding the actual in-control process mean being equal to the estimated in-control process mean), and the 75th percentile case (corresponding to *underestimation* of the in-control process mean $\hat{□}_0$). Let the true in-control rate parameter to be $□_0$= 2 with *dispersion* rate □ = 0.20. The mean and various percentiles of the sampling distribution of $\hat{□}_0$ are calculated for various samples and provided in Table 1. It is clear that these estimates deviate heavily from the true value $□_0$= 2 for samples ≤ *100*.

The conditional run length performance of the given chart is calculated for various parameters values, smoothing constant values and samples sizes. The estimated conditional run length distribution is provided in the Tables 2-4 for some choices. The various amounts of shifts in the average defects in terms of standard deviation of the in-control process are considered i.e. $\mu_a = \mu_0 + \delta\sqrt{\sigma_0}$. The case of $\delta = 0$ corresponds to the in-control performance and highlighted in bold. Several interesting observations can be made based on the run length performance summaries. We focus our discussion according to the value of δ.

Tables 2-4 indicate that the effect of parameter estimation is significant on the performance of the given chart. First consider the in-control performance (□ = *0*). ARL_0 and the corresponding SDRL values are very close in each case. ARL_0 is significantly deviated from expected (**500.00**), when the parameters are estimated from *m* initial samples. In the above tables, the nominal case refers to known parameters, so, the performance of the given control chart do not dependent on sample size. Underestimation cases (when parameters assume a value in the 75th percentiles) results in an increase in the number of false alarms and a decrease in the variability of the run length as compared to nominal summaries. On the other hand, overestimation cases (25th percentiles) results large EWMA variance than expected one. Therefore, the average run length ($□□□_0$) is expected to be less than the nominal values and in more frequent false alarms. Now consider the out-of-control scenario (□ > *0*). In Tables 2-4, the cases with favorable ARL results are the ones with smaller ARL_1 (out-of-control average run length) and larger ARL_0 values. From these tables it is clear that shift in the average nonconformities due to an assignable cause are detected faster for overestimation case that the underestimation as well as nominal.

It is apparent that the performance of the geometric-Poisson chart is significantly affected due to estimated parameters, however, the magnitude of the effect decreases as *m* increases. In both over or under estimation cases, a large reference sample size is required to achieve the desired in-control $□□□_0$ of 500.00. The choice of smoothing constant or EWMA weight has a significant effect on the performance of chart as it is obvious from Tables 2-4. Considering a sample size constant, it is observed that the actual $□□□_0$ value is decreased by increased the smoothing constant in case of underestimation while inverse hold for overestimation. To achieve the desired $□□□_0$ of 500.00 with a large reference sample size a large smoothing constant is required. However, when there exit a positive shift the average run length increases by increased in smoothing constant as it is obvious from Tables 2-4. Therefore, smaller smoothing constant is better than larger one in detecting small shifts of the average non-conformities.

Also, the dispersion parameter has significance effect on the performance of the given chart. The larger the *dispersion* parameter, the less effect on the □□□ when the *rate* parameter is fixed (see, Tables 2&4). This is due to the smaller value of *dispersio*n with fixed *rate* parameter converges to Poisson distribution. However, more the *rate* parameter with the fixed *dispersion* value yield less influenced on the performance (see, Tables 2&3.). Thus, choice of *dispersion* parameter □ has also significant effect on the performance of geometric-Poisson EWMA chart.

Marginal Performance of the geometric-Poisson EWMA chart

In practice, it is often not possible to know how the estimated mean compares to the true in-control mean. Therefore, it would also be useful to evaluate the marginal performance of a chart, chart, which considers the distribution of the estimated parameters to take into account the random variability introduced through parameter estimation. The marginal performance for the geometric-Poisson EWMA charts under different parameters values, sample sizes, smoothing constants and shift magnitudes. The ARL and SDRL for the given chart based on the estimated parameters for in-control $□_0$= 2.0 and $□_0$ = 0.20 and 0.30 values are calculated for in-control and out-of-control situations and given in Table 5. The corresponding ARL for $□_0$= 3 and $□_0$ = 0.20 for different smoothing constant values are given in Fig.1 (a)-(c), which are known as ARL curves. The run length characteristic for any other combination of the parameters could be obtained similarly.

In the Table 5 and Figures 1 (a)-(b), the performance metrics are weighted averages over all the values that the estimation may yield for the in-control parameters. To study how large the sample size should be to perform essentially like the known parameter case, the values 30, 100, 1000, and ∞ for *n* are evaluated. The infinite sample size ($n = \infty$) corresponds to the known parameters or the nominal case.

Comparing the values in Table 5 and Fig.1 (a)-(b) with their nominal values, it is obvious that estimating control limits can cause both ARL and SDRL to be large than their desired values when the process is working in-control, especially for smaller smoothing constant vales and smaller sample sizes. For a fixed sample size and dispersion parameter, the chart produces the in-control ARL large as the EWMA smoothing constant increases. But for increasing sample size $n \geq 100$, the ARL_0 converges to 500.00 as expected. When a shift in the average number of defects occurs, a large sample $n \geq 100$ and larger smoothing constant value is required to have a reasonable ARL_1 performance. In general, the effect of the sample size on the marginal performance of the control chart is more for smaller choices of *W*, and less for larger choices of *W*. Also, the effect of the sample size is most significant for the smaller mean shift. The choice of smoothing constant and sample size would depend on how fast one wants to detect a shift of given size in average non-conformities. However, as a rule of thumb we recommend sample sizes of at least 500 to achieve better detection of process shifts. Furthermore, smoothing parameter values W = 0.05 or 0.10 may also be suggested to detect small-sized shifts. Also, the larger value of *dispersion* parameter would be required to minimize false alarms. The similar behavior has been observed for other choices of parameters and smoothing constant.

Illustrative Example

In this section, we provide an illustrative example to demonstrate the practical implementation of the under study estimated control limits. We have generated a data set of 100 observations from the geometric-Poisson distribution with parameters, $□ = 2.5$ and $□ = 0.20$ by following Chen (2012). The control limits of the geometric-Poisson EWMA chart are estimated based on (i) first 30 observations and (ii) all 100 observations. The smoothing constant value w = 0.10 and control chart multiplier value A = 2.75 are used.

The in-control $\hat{□}$ estimates are 3.10 for 30 observations and 3.45 for 100 observations, and $\hat{□}$ estimates are 0.80 for 30 observations and 0.50 for 100 observations respectively. The generated samples are used to calculate the geometric-Poisson EWMA statistic and the estimated control limits. The geometric-Poisson EWMA control chart using this information is plotted in Figure 2 (a-b) respectively. The both charts reveal that the process is on-control. Notice the difference between the widths of the control limits of the two charts. Although the case of 100 observations is expected to perform in terms of in-control ARL approximately as designed, this will not be true for the case of 30 observations because it required smaller than expected in-control ARL because of the tighter control region.

CONCLUSION AND DISCUSSION

The Poisson distribution is often used to model the count data in all fields. However, the Poisson distribution is not only underlying distribution for counting data. For production processes, the geometric-Poisson EWMA control chart, proposed based on geometric-Poisson compound distribution, is very useful to detect the process variation rapidly to reduce the lost cost. This chart could be used and should be used if small shifts from normal conditions are important to detect quickly.

In real application, actual values of process parameters for designing the given chart are often unknown. In this situation, a typical approach is to conduct a Phase-I study, where a reference sample of *m* observations is obtained and then used for estimating these unknown parameters. However, the performance of the control chart may significantly be different than expected performance if the parameters are not well estimated. This article investigates the performance of the geometric-Poisson EWMA chart when the process parameters are estimated based on *m* reference samples. The effect on the run length characteristics such as ARL and SDRL has been shown to be significant. Furthermore, for smaller EWMA smoothing constants, say 0.05, the chart with estimated parameters produces more false alarm rate which results into large in-control ARL and SDRL than the chart with known parameters. This study suggest minimum 500 sample size and smoothing constant greater than 0.05. However, this choice depends on the sensitivity of the chart with respect to detecting changes in average non-conformities. The larger value of *dispersion* parameter is better to get the desired in-control ARL and SDRL. The results of the study are very useful for practioners and researchers to design a geometric-Poisson EWMA chart for detecting minor process variations in production processes and improving the process quality in Phase I sample.

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